The fast Fourier transform

A simple matrix factoring example is used to intuitively justify the FFT algorithm. The matrix formulation requires us to consider the discrete Fourier transform, given an array of perhaps complex numbers, x_n (n=0, 1, 2,....N-1) and X_m (m=0, 1, 2,....M-1) where,

$$X_n = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i m_n}{P}}$$

set,

$$k = mT$$

$$n = \frac{n}{PT}$$

We note that this describes the computation of N equations, let W be the sum of each term. This can be written as,

$$\begin{aligned}
X_0 &= x_0(0)W^0 + x_0(1)W^0 x_0(2)W^0 x_0(3)W^0 \\
X_1 &= x_0(0)W^0 + x_0(1)W^1 x_0(2)W^2 x_0(3)W^3 \\
X_2 &= x_0(0)W^0 + x_0(1)W^2 x_0(2)W^4 x_0(3)W^6 \\
X_3 &= x_0(0)W^0 + x_0(1)W^3 x_0(2)W^6 x_0(3)W^9
\end{aligned}$$

Or in matrix form,

$$\begin{vmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{vmatrix} = \begin{vmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^1 & W^2 & W^3 \\ W^0 & W^2 & W^4 & W^6 \\ W^0 & W^3 & W^6 & W^9 \end{vmatrix} \cdot \begin{vmatrix} x_0(0) \\ x_0(1) \\ x_0(2) \\ x_0(3) \end{vmatrix}$$

Or more concisely as,

$$X(n) = W^{nk} x_0(k)$$

The FFT owes its success to the fact that the algorithm reduces the number of multiplication's and additions required in the computation of the above expression. A full proof of the FFT is available in Chapter 11 of *"The fast Fourier transform"*, by E. O. Brigham.

Problems associated with the FFT algorithm:

A mathematical proof of the Fast Fourier Transform, FFT, algorithm is beyond the scope of this discussion, however, an excellent explanation of such can be found *in "The Fast Fourier Transform"* by E. O. Brigham, please see references. There are three phenomenon associated with the discrete Fourier transform.

1. The first is also associated with the continuous Fourier transform that is time scale expansion, this also corresponds to frequency scale compression. It is important to recognise that the amplitude increases to hold the area constant.
2. The effect known as *"truncation"* can be seen in more of the results, to some degree at least, that have been obtained from the FFT. This effect, which occurs at other than a multiple of the period of the oscillation, has the result of creating a periodic function with sharp discontinuities. These sharp discontinuities cause, when transformed, additional frequency components in the frequency domain, i.e. *"spectral colouration"*. The frequency function no longer remains as a single impulse but is converted into a continuous function of frequency with a local maximum. The maximum is still located at the original impulse. The other peaks are responsible for the leakage in the discrete Fourier transform.
3. Leakage can be reduced by employing a time domain truncation function which has *"Side-lobe"* characteristics of smaller magnitude than those of the sine function.. Accuracy of results can be improved by decreasing T, the sample interval and increasing N, the number of samples.