

Use of the Generalized Addition Theorem for Spherical Waves for the Reflection Suppression by Spherical Mode Filtering

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Abstract—This paper presents the results of a recent study into the use of the generalized vector addition theorem for antenna position translation for the purposes of orthogonalizing and then extracting the deleterious effects of parasitically coupled scatterers in spherical antenna measurements directly from the computed spherical mode expansion without the need to transform to the asymptotic true far-field in order to implement the all-important antenna translation. The generalised vector addition theorem can be successfully utilised to perform antenna position translations in any direction, and for displacements that can be smaller *or* larger than the maximum radial extent of the antenna. The revised algorithm represents a notable development as it is rigorous and general incorporating both reactive and propagating components, thereby making the processing applicable to a wider range of problems than has previously been the case. Results are presented that validate and illustrate the effectiveness of the new algorithm.

Index Terms—addition theorem, mode filtering, reflection suppression, spherical mode expansion

I. INTRODUCTION

Mode filtering based monochromatic scattering suppression technique have been shown to be very applicable for many different forms of near-field [1, 2, 3, 4, 5, 6], far-field [7, 8] antenna measurements, and RCS measurements, with recent advances extending the benefits of compressive sensing and sparse sampling [9, 10] to the technique. Usually, the technique takes a far-field pattern, either measured directly or obtained using an appropriate near-field to far-field transformation. The measurement is required to be conducted such that the antenna under test (AUT) is positioned at an offset with respect to the measurement coordinate system origin. This offset introduces a phase taper over the AUT pattern and results in a far greater interference between the direct and parasitically coupled, spurious, scattered signals. When the AUT far-field pattern including this offset is expanded using the spherical wave expansion technique [11], it can be seen that the resulting spherical mode coefficients of the AUT are distributed over large number of N -indices, with N_{\max} depending largely on the radius of a conceptual sphere which includes the majority of the current sources that contribute to the far-field radiation pattern, and which is also centred about the origin of the measurement coordinate system [12]. It can then be shown that when the AUT is mathematically displaced back to the origin, which reduces the phase taper on the corresponding far-field pattern, the

number of N -indexed spherical mode coefficients required to describe the AUT far-field pattern reduces together with a corresponding reduction in the size of the circumscribed sphere which includes all significant AUT contribution sources [13]. On the other hand, and due to this same displacement, the spherical mode coefficients associated with the extraneous parasitically coupled sources move towards higher order N - and M -indexed spherical modes. Due to this opposite directed migration of the spherical mode coefficients (SMC) of the AUT, towards lower-order indices, and the extraneous sources, towards higher-order indices, a filter can be applied to effectively remove the higher-order modes and thus suppressing the energy of the extraneous sources within the measurement [13, 14].

This well-established mode-filtering based approach relies on the derived far-field pattern which can, in principle, be obtained using any type of far- or near-field measurement technique, and it can therefore be seen as being generally applicable [14]. However, for the special case of the spherical near-field measurement approach, it is a little cumbersome to first perform a probe-corrected spherical near-field to far-field transformation, which itself involves the computation of the spherical mode coefficients, and then after the displacement has been applied to the far-electric-fields, to perform another spherical wave expansion and summation to implement the mode filtering step [13, 14]. While this approach has been widely utilised and exhaustively validated, it comes at the expense of additional computational effort and can result in some loss of information due to the requirement to pass through the asymptotic far-field which can therefore impose limitations on further near-field applications such as near-field pattern extraction [4] or antenna diagnostics [6]. However, in this paper, an alternative algorithm is presented where the displacement of the AUT is performed directly on the SWCs using the generalised addition theorem for spherical waves [15] for which an efficient, numerically stable, implementation has been described in [16, 17]. It will be shown that the new rigorous algorithm for the translation and filtering can be easily implemented directly within the standard data processing chain of any spherical near-field transformation algorithm avoiding the need to first transform to the asymptotic far-field, and also removing the need for a secondary spherical mode expansion and secondary spherical mode summation.

While the addition theorem required for the spherical near-field to far-field transformation algorithm has been described in detail [11], its implementation is both limited to the case of impinging waves and positive z -directed translations only. In the presented paper the addition theorem will be derived in a new form [15] which allows for the translation in any desired direction, as well as being valid for solutions for waves transitioning through the sphere and for the case where the magnitude of the translation is smaller than the radius of the minimum sphere. Furthermore, it will be shown its wide applicability to spherical near-field measurements and especially in the application for reflection suppression [14]. The next section presents an overview of the translation of origins by generalised vector addition before the remaining sections present preliminary results. The paper finishes with a summary and conclusion.

II. ISOMETRIC TRANSLATION OF ORIGINS BY GENERALISED ADDITION THEOREM

The effectiveness of the frequency-domain, spherical mode-orthogonalization and filtering reflection suppression technique is achieved by enforcing a separation of modes in the spherical multipole expansion through the use of a suitable translation operator [13, 14]. Traditionally this isometric coordinate translation was implemented in the asymptotic, *i.e.* true, far-field by virtue of the multiplication of the far electric fields by a complex exponential function that corresponded to applying a differential phase change that represents a physical translation of the test antenna [13]. Since it is well-known that a multiplication in one domain can correspond to a convolution in another domain, *i.e.* the spherical multipole expansion, where the convolution operator can be seen to be represented by a linear summation. Thus, as we shall present, it is perhaps unsurprising that the generalised vector addition theorem corresponds to a linear “summation” operation. Motivated by this, let us first consider two conventional right-handed coordinate systems where we assume that these are initially coincident and synonymous. We now assume that the unprimed system is fixed in space, with the primed system being displaced by a distance A in the direction specified by the conventional right-handed polar spherical angles θ_0, φ_0 . It can be shown that the translation of spherical waves can be expressed, when using the shorthand formulation of [18], as a linear operation [19] so that the spherical wave function $\vec{F}_{smn}^{(c)}(r, \theta, \varphi)$ can be expressed as a combination of spherical waves defined in the primed system $\vec{F}_{\sigma\mu\nu}^{(c)}(r', \theta', \varphi')$ as [17],

$$\underline{E}_{smn}^{(c=3)}(r, \theta, \varphi) = \sum_{\sigma\mu\nu} C_{\sigma\mu\nu}^{(c=1)}(\theta_0, \varphi_0, k_0 A) \underline{E}_{\sigma\mu\nu}^{(c=3)}(r', \theta', \varphi') \quad (1)$$

Where the summation is understood to be taken over,

$$\sum_{\sigma\mu\nu} = \sum_{\sigma=1}^2 \sum_{\nu=1}^{\infty} \sum_{\mu=-\nu}^{\nu} \quad (2)$$

Here $\vec{F}_{\sigma\mu\nu}^{(c)}(r', \theta', \varphi')$ are a set of spherical vector wave functions inside the primed coordinate system, and $C_{\sigma\mu\nu}^{smn(c=1)}(\theta_0, \varphi_0, kA)$ the translation coefficients with (c) indicating the spherical radial function assuming the usual 1,2,3 and 4 numbering system, *cf.* [18]. Thus, we use $c = 3$ for the

spherical vector wave functions before, $\vec{F}_{\sigma\mu\nu}^{(c)}(r', \theta', \varphi')$, and then after, $\vec{F}_{smn}^{(3)}(r, \theta, \varphi)$ in the unprimed coordinate system once the translation operation has been applied, matching the outward travelling wave assumption [11]. This is a very general statement since it may be shown that this is equally valid for inward travelling waves where $(c) = 4$. In this case, we use $(c) = 1$, *i.e.* Bessel functions of the first kind within the calculation of $C_{\sigma\mu\nu}^{smn(c=1)}(\theta_0, \varphi_0, kA)$ so that this translation can be applied to either incoming or outgoing waves and where equation (1) is a generalization of the far-field differential phase change operation described above, and is equally valid for large magnitudes of A . Where A is the distance of the physical offset applied to the antenna. It is noted here that open literature different normalizations factors have been applied to the generating spherical wave functions and thus the same normalization needs to be applied translation coefficients. While the present work assumes a normalization as described in [18], the calculated translation coefficients using [19] need to be renormalized by a factor of,

$$\alpha = \sqrt{\frac{\nu(\nu+1)}{n(n+1)}} \quad (3)$$

Considering that the minimum radial extent of the translated antenna also increases with $r + A$, thus the translation matrix also needs to be extended to ensure that sufficient higher order modes are included to correctly represent the translated antenna [17]. The calculation of the translation coefficients $C_{\sigma\mu\nu}^{smn(c)}(\theta_0, \varphi_0, kA)$ are as defined in the open literature [17, 19] with the details being beyond the scope of the present work. However, it is important to note that this is a fundamentally *different* operation to the translation required for probe compensated spherical near-field measurements, where both far-field phase *and* amplitude functions are affected by the translation operation [17]. Figure 1 below presents a schematic representation of the isometric translation of origins operation embodied by equation (1). Having now established the translation operation, the next section presents the results of the utilisation of this approach.

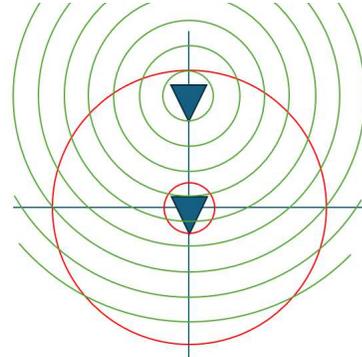
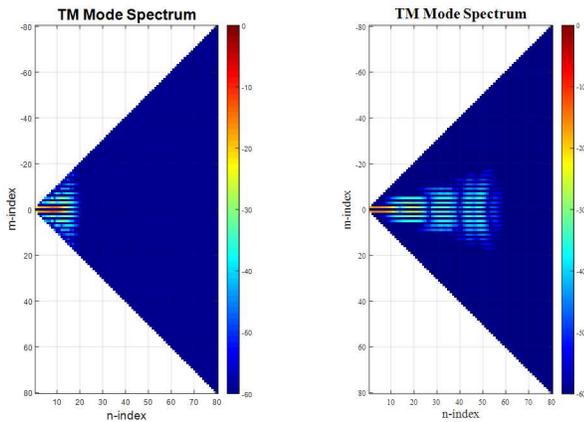


Figure 1: Translation of origin of the antenna as implemented by equation (1). Here, the red circle denotes the antenna’s minimum radial extent before, and then after the translation. The green circles represent the phase fronts after translating the antenna.

III. VERIFICATION OF TRANSLATION OF ORIGIN FORMULA

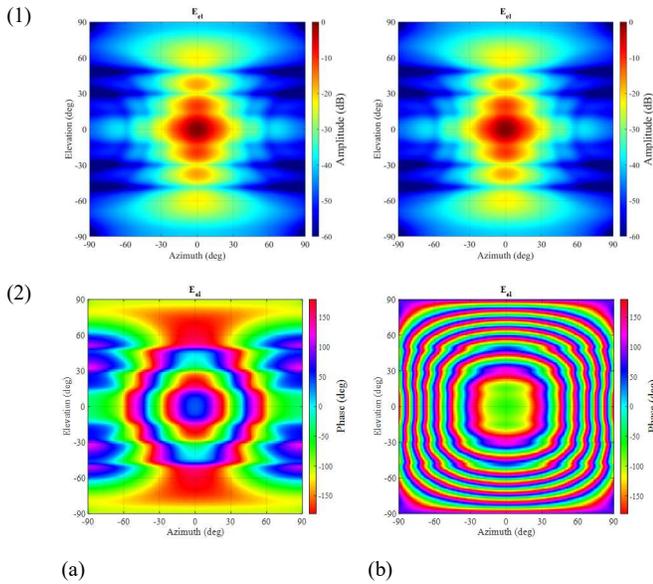
By way of a preliminary verification, a rectangular pyramidal horn pattern was simulated using the field transformation of its y -axis polarised aperture illumination

function. The Kirchoff-Huygens formula [13] was used to produce the full-sphere, far electric fields. The far electric fields were then expanded onto a set of spherical modes, *cf.* Figure 2a. Here, the lowest *circa* 25 modes contain the majority of the radiated power and are sufficient to reliably represent the antenna far-field pattern since $n_{max} = k_0 r + 10 = 25$. Then by applying the translation coefficient, calculated using an offset (A) of $z=0.3$ m, the resulting spherical mode distribution following the application of the translation operation can be seen presented in Figure 2b. When comparing Figures 2a and 2b, it is clear that the occupied mode space in N can be seen to have significantly increased, extending up to *circa* $n = 60$ modes. This is expected since $n_{max} = k_0(r + A) + 10 = 77$. Note, the inclusion of the additional ten modes is an arbitrary safety factor that depends upon stability and the accuracy required, and which can be larger in cases where very many modes are required, *cf.* [13].



a) Before translation. b) After translation.

Figure 2: Mode Spectra plot of simulated rectangular pyramidal horn before, a) and after, b) antenna translation in $z=0.3$ m direction.

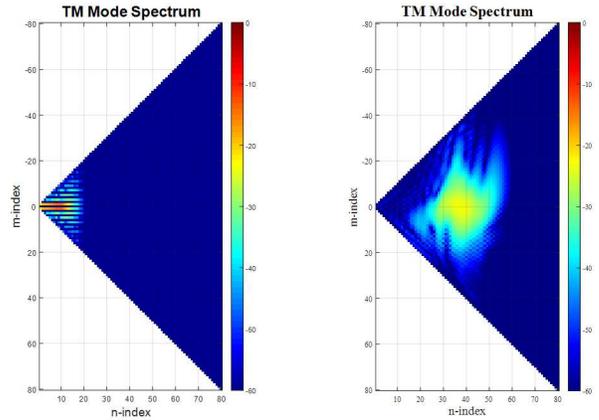


(a) (b)

Figure 3: Far-field copolar amplitude (1) and phase (2) plot of simulated rectangular pyramidal horn comparing (a) before translation (b) after translation when the translation was implemented in the SMC domain.

Figure 3 presents plots of the equivalent copolar far-fields where the far-fields equating to the mode plot shown in Figure 2b can be seen compared with the far-field obtained from the conventional differential phase change [13]. As expected, the far-field amplitude functions are identical. The difference to the far-field function is only manifest in the phase function which can be seen presented in the second line (2) of Figure 3. Similarly encouraging agreement was attained in the case of the cross-polar component however it not included here as a consequence of available space.

While the described translation coefficients as defined in (1) is not limited to any asymptotic translation along the z -axis only, translations can also be applied in any other direction. To verify this a displacement in $(x=0.1, y=0.2, z=0.1)$ m is performed next using the same simulation of a SGH as a base.

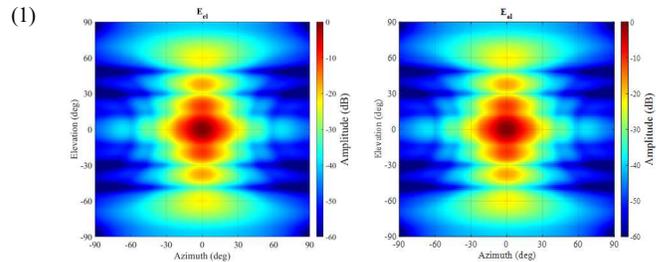


a) Before translation. b) After translation.

Figure 4: Mode Spectra plot of simulated rectangular pyramidal horn before, a) and after, b) antenna translation in $(x=0.1, y=0.2, z=0.1)$ m direction.

In Figure 4 the mode spectra of the non-translated case is compared to the translated case. As opposed to Figure 3, in Figure 4 it becomes apparent that as expected the translation does not only redistributes the mode energy over higher order N -modes of constant m -index, but also a redistribution across the M -spectrum is found. While still the total number of approximately 60 modes are expected to be sufficient describe the antenna (up to 80 are plotted).

In Figure 5 the resulting far-field plots are shown of both the baseline and the translated case. While comparing the phase distribution plots (2) it is apparent that the translation is in the approximate direction of Azimuth = Elevation = 60 degrees, which matches the direction calculated from the xyz-offsets.



(1)

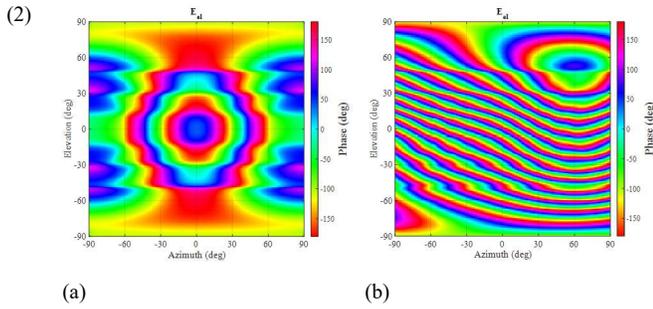


Figure 5: Far-field copolar amplitude (1) and phase (2) plot of simulated rectangular pyramidal horn comparing (a) before translation (b) after translation when the translation was implemented in the SMC domain.

As the functionality of the SMC translation operation has been established, we are free to utilise this for the purposes of scattering suppression which we consider in the next section.

IV. RESULTS OF REFLECTION SUPPRESSION BY GENERALISED ADDITION THEOREM

To further validate the use of the generalized vector addition theorem for reflection suppression, a full-wave, three-dimensional, computational electromagnetic (CEM) simulation was created in FEKO [20] and used to model a conventional polar-mode SNF acquisition with, and without a large scattering body. Details of the CEM simulation can be found in [10, 17] and only a summary is presented here. A model of a 3.5 GHz, 48-element, rectangular, linearly polarised, finite array antenna was used as a test AUT in a conventional SNF polar-mode acquisition using a “model tower” ϕ over θ spherical positioning system with an x -axis copolar orientation. The AUT was excited by an ideal network with equal amplitude and phases at each of the input ports. Although the precise excitation is not a matter of concern for this investigation, the complex nature of the field distribution renders it a far more attractive candidate than a simple antenna model such as that which was used in the preceding section. The simulated measurement is shown in Figure 6. Here, the AUT was displaced by 360 mm from the origin of the simulated spherical “measurement” coordinate system, which is an amount that was comparable with the maximum radial extent of the un-displaced AUT. This is a magnitude that is typically thought sufficient to allow for the separation of antenna modes and parasitically coupled modes [13]. This radiator was then perturbed with the inclusion of a stationary perfect electric conducting (PEC) 400 mm \times 400 mm flat square plate that was used to create an electromagnetically complex, test environment. This was an especially large reflector since it was larger than the array antenna. The advantages of this approach are that: the true unperturbed far-field antenna pattern is known precisely, the position of the scatterer can be changed freely, and we know precisely what the perturbing factors are; unlike in an actual range measurement where many other factors are also present [13].

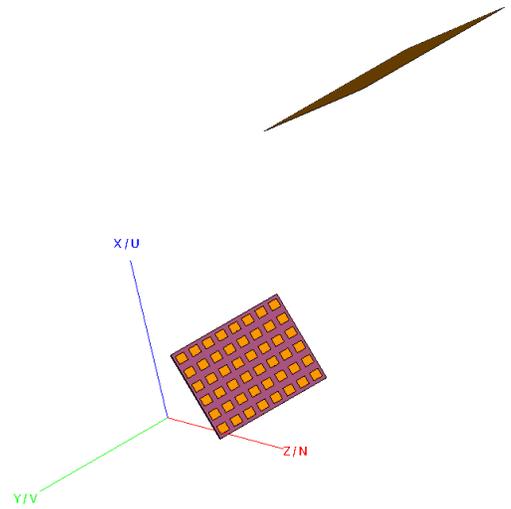
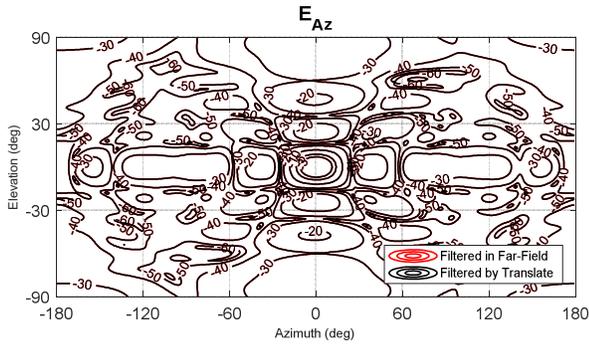
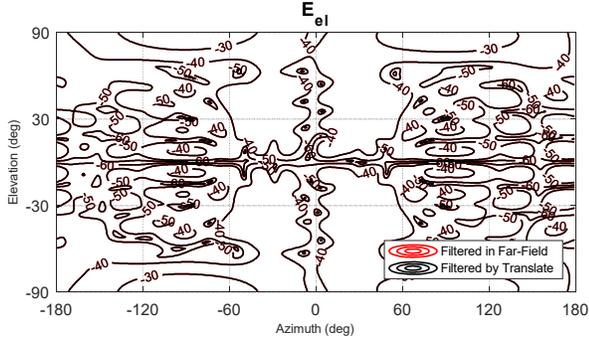


Figure 6: The “measurement” setup, the AUT is on the left, and the scatterer is on the top of the figure..

With this CEM simulation to hand, the revised mode-filtering algorithm could be further verified by means of a comparison with the traditional mode-filtering algorithm [2, 3, 13, 14]. While the new method in itself is expected only to attain significant benefits in those cases where the evanescent modes of the antenna in the near-field become more significant or when attempting to extract isolated single element radiating properties from array antenna measurements, which, it is worthwhile to note, was one of the early applications of spherical mode filtering [4], it is none the less important to verify equivalence with the traditional approach within its region of validity. It is important to note that the same SNF data, physical displacement, mode cut-off, and spherical mode filter function were used within each of the respective data processing algorithms so that differences observed in the resulting far-field patterns would be purely attributable to the algorithms themselves rather than being associated with some other factor. Thus, this SNF data set was processed using the conventional and new algorithms with the results being presented in Figure 7 in the form of overlaid iso-level, *i.e.* contour, plots. Here, the red trace denotes the results from the standard reflection suppression algorithm, while the black contours denote the result obtained from the new algorithm. Figure 7a presents the copolar, Ludwig II Azimuth polarised field component whilst Figure 7b contains the cross-polar Elevation polarised component [13]. Here, the degree of agreement attained is attested to by the fact that the red contours, which were plotted under the black contours, are not visible. Thus, one may observe that the results of both filtering methods provide a near identical result. A single quantitative objective measure of adjacency was determined from the RMS dB difference level [13] which was found to be on the order of -300 dB, which is at the limit of double precision arithmetic, and indicating effective equivalence between the respective algorithms for this case which was largely free of reactive field components.



a) Far-field copolar amplitude comparison plot.



b) Far-field cross-polar amplitude comparison plot.

Figure 7: Far field copolar, a) and cross-polar, b) pattern obtained from far-field based translation mode filtering compared to spherical wave translation based mode filtering presented tabulated on an azimuth over elevation coordinate system and resolved onto an azimuth over elevation polarization basis.

V. CONCLUSIONS

This paper introduces a novel algorithm that is a generalised version of the traditional, frequency domain, spherical mode filtering based, scattering suppression technique that is exactly equivalent to that routine in the limit where reactive, evanescent, fields are absent but that no longer requires spherical near-field data to be transformed to the asymptotic true far-field to implement the requisite translation of origins. This study exploited the existence of an extensive full-wave CEM simulation to provide the necessary data required to enable the very precise verification and validation of the new algorithm. The generalised spherical addition theorem was used to implement the AUT isometric translation, in any direction, and by a displacement that could be larger, smaller, or equal to the MRE. The translation operation was applied directly to the antenna SMCs and is entirely general since it is *not* limited to considering just the first order, $m = \pm 1$, azimuthal modes, but rather is valid for *all* azimuthal modes for which $|m| \leq n$. The success of the revised, streamlined, spherical mode filtering bases reflection suppression algorithm is attested to by a comparison with a traditional algorithm with agreement being on the order of double precision arithmetic for this special case. This is crucial as it provides valuable verification for both algorithms and is the first time that this validation has been reported within the open literature. The planned future work is

to include further verification of the translation operation when applied in directions other than along the z -axis.

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